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COLOURING SERIES–PARALLEL GRAPHS

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We establish a minimax formula for the chromatic index of series–parallel graphs; and also prove the correctness of a "greedy" algorithm for finding a vertex-colouring of a series–parallel graph.

1. Introduction

In this paper we prove two results, which are not related to one another except that they both involve colouring series–parallel graphs. The easier of the two concerns the correctness of a "greedy algorithm" for finding a vertex-colouring of a series–parallel graph, and we postpone all further discussion of this to Section 6. The other result concerns edge–colourings, and will be dealt with in Sections 1–5.

A *k-edge-colouring* of a graph G is a map $\kappa : E(G) \to \{1, \dots, k\}$ such that for distinct edges e, f if $\kappa(e) = \kappa(f)$ then e and f have no common end. (Graphs in this paper are finite, and may have multiple edges but not loops; $V(G)$ and $E(G)$ denote the vertex- and edge-sets of a graph G.) The *chromatic index* $\chi'(G)$ is the minimum $k \geq 0$ such that G has a k-edge-colouring. The problem of determining $\chi'(G)$ is NP-hard; indeed, deciding whether a 3-connected cubic graph G satisfies $\chi'(G) = 3$ or not is NP-complete [6]. On the other hand, for planar graphs G it is not known whether determining $\chi'(G)$ is NP-hard; and indeed for planar graphs there is a conjectured minimax formula (which, if true, would be convertible into a polynomial algorithm, via [3,4]) which we now discuss.

Let $\Delta(G)$ denote the maximum valency of the vertices of G (the *valency* of a vertex is the number of edges incident with it). Then clearly $\chi'(G) \geq \Delta(G)$, but equality need not occur (for example, when $G = K_3$). There is, however, another useful lower bound on $\chi'(G)$, as follows. For $X \subseteq V(G)$, we denote by $\overline{X}$ the set of edges of G with both ends in X. In any $\chi'(G)$-edge-colouring κ of G, at most $\lfloor |X|/2 \rfloor$ edges in $\overline{X}$ have the same colour (where $\lfloor p \rfloor$ denotes the greatest integer n with $n \leq p$), and hence $|\overline{X}| \leq \chi'(G) \cdot \lfloor |X|/2 \rfloor$. This follows from our other bound

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$\chi'(G) \geq \Delta(G)$ if $|X|$ is even, but if $|X|$ is odd (and at least 3) it sometimes provides new information. Let

$$\Gamma(G) = \max\left(\frac{2|\overline{X}|}{|X|-1} : X \subseteq V(G),\ |X| \geq 3 \text{ and odd}\right).$$

We have shown then that $\chi'(G) \geq \max(\Delta(G), \lceil\Gamma(G)\rceil)$ (where $\lceil p \rceil$ means $-\lfloor -p \rfloor$) for every graph G. It was conjectured in [10] (and remains open) that

(1.1) (Conjecture). *If G is planar then $\chi'(G) = \max(\Delta(G), \lceil\Gamma(G)\rceil)$.*

Equality does not hold for graphs in general; for instance if G is the Petersen graph, or that graph minus one vertex, then $\chi'(G) = 4$ but $\Delta(G) = \Gamma(G) = 3$. Conjecture (1.1) is no doubt difficult, if true, because it contains the four-colour problem (for it implies that 3-connected cubic planar graphs are 3-edge-colourable). Our object here is to prove Conjecture (1.1) for series–parallel graphs. (Please note that for *simple* series–parallel graphs this is not very difficult; but we wish to prove it permitting multiple edges.) Marcotte [7] showed that $\chi'(G) \leq \max(\Delta(G)+1, \lceil\Gamma(G)\rceil)$ for outerplanar graphs G, and has recently [8] shown the same thing for series–parallel graphs and also for a more general class of graphs.

A graph is *series–parallel* if it can be reduced to the null graph by repeated use of the following three operations:

(i) delete an edge parallel with another edge

(ii) contract an edge in series with another edge (that is, incident with a vertex of valency 2, and not parallel with the second edge at this vertex)

(iii) delete a vertex of valency ≤ 1.

(This is not quite the standard definition because we are excluding loops, and because frequently series–parallel graphs are assumed to be 2-connected.) It was proved in [1,2] that

(1.2) *G is series–parallel if and only if no subgraph of G is a subdivision of K_4.*

[K_4 is the complete graph with 4 vertices; and G is a *subdivision* of H if G is reducible to H by repeated use of operation (ii) above.]

Until the end of Section 5, let $k \geq 0$ be some fixed integer. If G is a graph and $X \subseteq V(G)$, we define $\delta(X) = \delta_G(X) = k|X| - 2|\overline{X}|$. We say that X is *odd* if $|X|$ is odd, and *even* otherwise.

(1.3) *$\Gamma(G) \leq k$ if and only if $\delta(X) \geq k$ for every odd $X \subseteq V(G)$.*

Proof. Let $X \subseteq V(G)$ be odd. Then $2|\overline{X}| \leq k(|X|-1)$ if and only if $k|X| - \delta(X) \leq k(|X|-1)$, that is, $\delta(X) \geq k$. Since $\delta(X) = k$ if $|X| = 1$, the result follows. ∎

We shall prove

(1.4) *Let G be a series–parallel graph with $\Delta(G) \leq k$, such that $\delta(X) \geq k$ for every odd $X \subseteq V(G)$. Then $\chi'(G) \leq k$.*

In view of (1.3), this will prove Conjecture (1.1) for series–parallel graphs.

2. Colouring rooted graphs

We denote the valency of a vertex v of a graph G by $d(v)$ or $d_G(v)$ when there is a danger of ambiguity. A graph G is *free* if $\Delta(G)$, $\Gamma(G) \leq k$; that is, by (1.3), if

(i) $d(v) \leq k$ for each $v \in V(G)$ and
(ii) $\delta(X) \geq k$ for each odd $X \subseteq V(G)$.

A *rooted graph* is a triple (G, s, t), where G is a graph and $s, t \in V(G)$ are distinct. We define $p(G, s, t)$ (or $p(G)$, when there is no danger of ambiguity) to be the minimum value of $\delta(X)/2$ taken over all even $X \subseteq V(G)$ with $s, t \in X$. (This is well-defined, since $\{s, t\}$ is even.) Similarly we define $q(G, s, t)$ (or $q(G)$) to be the minimum of $(\delta(X) - k)/2$, taken over all odd $X \subseteq V(G)$ with $s, t \in X$, if such an X exists, and otherwise $q(G, s, t) = k$.

(2.1) *$p(G), q(G)$ are integers, and if G is free then $p(G), q(G) \leq k$.*

Proof. By definition, $\delta(X) = k|X| - 2|\overline{X}|$. Thus if X is even then $\delta(X)$ is even, and if X is odd then $\delta(X) - k$ is even, and so $p(G), q(G)$ are integers. Since $\{s, t\}$ is even and $\delta(\{s, t\}) \leq 2k$, it follows that $p(G) \leq k$, and similarly that $q(G) \leq k$ (by choosing some third vertex u of G and considering $\delta(\{s, t, u\})$).

(2.2) *If G is free then $p(G) \geq k - d(s)$.*

Proof. Choose $X \subseteq V(G)$, even, with $s, t \in X$ and $p(G) = \delta(X)/2$. Then $X - \{s\}$ is odd and so $\delta(X - \{s\}) \geq k$ since G is free; but $\delta(X) \geq \delta(X - \{s\}) + k - 2d(s)$ and the claim follows. ∎

If X, Y are sets, we define $X \Delta Y = X \cup Y - (X \cap Y)$.

(2.3) *If G is free then $p(G) + q(G) \geq 2k - d(s) - d(t)$.*

Proof. The result holds if $|V(G)| = 2$, and we assume that $|V(G)| \geq 3$. Choose $X, Y \subseteq V(G)$ with $s, t \in X, Y$ and with X even and Y odd, such that $p(G) = \delta(X)/2$ and $q(G) = (\delta(Y) - k)/2$. Thus

$$\delta(X) + \delta(Y) = 2p(G) + 2q(G) + k.$$

Let r be the number of edges of G with one end in $X \cap Y$ and the other in $X \Delta Y$. Then

$$2\delta(X \cap Y) + \delta(X \Delta Y) - 2r \leq \delta(X) + \delta(Y).$$

Moreover, $\delta(X \Delta Y) \geq k$ since $X \Delta Y$ is odd and G is free, and it follows that

$$\delta(X \cap Y) - r \leq p(G) + q(G).$$

On the other hand,

$$k|X \cap Y| - \delta(X \cap Y) + r = 2|\overline{X \cap Y}| + r \leq \sum_{v \in X \cap Y} d(v) \leq k(|X \cap Y| - 2) + d(s) + d(t)$$

that is,

$$\delta(X \cap Y) - r \geq 2k - d(s) - d(t).$$

The result follows. ∎

(2.4) *If G is free then $q(G) \geq k - d(s) - d(t)$ and $q(G) \geq 0$.*

Proof. The first follows from (2.1) and (2.3). For the second, let $X \subseteq V(G)$ be odd, with $s, t \in X$. Then $\delta(X) \geq k$ since G is free, and so $(\delta(X) - k)/2 \geq 0$. The result follows. ∎

A *goal* for a rooted graph (G, s, t) is a pair (S, T) of subsets of $\{1, \ldots, k\}$. It is *attained* by (G, s, t) if there is a k-edge-colouring κ of G such that for each $e \in E(G)$, if $e \sim s$ then $\kappa(e) \notin S$, and if $e \sim t$ then $\kappa(e) \notin T$. [We write $e \sim s$ if e is incident with s.] We say that a goal (S, T) is *reasonable* if $|S \cup T| \leq p(G, s, t)$, $|S \cap T| \leq q(G, s, t)$, $|S| + d(s) \leq k$ and $|T| + d(t) \leq k$. It is easy to show that if (G, s, t) attains a goal (S, T) then G is free and (S, T) is reasonable. (We omit the proof since we shall not need the result.) What will concern us is the reverse implication. Let us say that (G, s, t) is *successful* if G is free and (G, s, t) attains every reasonable goal. Our objective is to develop ways to make larger successful rooted graphs by piecing together smaller ones, and these will in effect allow us to construct all free series–parallel graphs from trivial ones.

3. Two integer programmes

Before we begin combining successful graphs, we prove two lemmas for later use.

(3.1) *Let a, b, c, d, m, n, p, q, r, s be integers. There are integers $x, y, z, w \geq 0$ such that $(x, y, z, w)A \leq (a, b, c, d, m, n, p, q, r, s)$, where*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix},$$

if and only if all the following 26 quantities are non-negative: a, b, c, d, $m+n$, $a+s$, $b+p$, $c+q$, $d+r$, $a+r+s$, $b+p+s$, $c+p+q$, $d+q+r$, $a+c+n$, $b+d+m$, $a+n+r$, $b+m+s$, $c+n+p$, $d+m+q$, $n+p+r$, $m+q+s$, $a+q+r+s$, $b+p+r+s$, $c+p+q+s$, $d+p+q+r$, $p+q+r+s$.

Proof. Let G be the directed graph of Figure 1, where each edge e is assigned a capacity $c(e)$ as indicated. Now the desired x, y, z, w exist if and only if there is an integer-valued function ϕ with domain $E(G)$, satisfying

(i) for each vertex v, $\sum_{e \in A} \phi(e) = \sum_{e \in B} \phi(e)$, where A and B are the sets of edges entering and leaving v

(ii) for each edge e, $\phi(e) \leq c(e)$.

(Let x, y, z, w be $\phi(e)$ where e is edge with capacity a, b, c, d respectively.) By Hoffman's circulation theorem [5], such a function ϕ exists if and only if for every partition (X, Y) of $V(G)$, if there is no edge of G from Y to X then $\Sigma c(e) \geq 0$ where the sum is taken over all edges from X to Y. By enumerating all such partitions, the result follows. ∎

(3.2) *Let $a_1, a_2, b_1, b_2, c_1, c_2, d, e, f$ be integers. There are integers $x_1, x_2, y_1, y_2, z \geq 0$ such that $(x_1, x_2, y_1, y_2, z)A \leq (a_1, a_2, b_1, b_2, c_1, c_2, d, e, f)$, where*

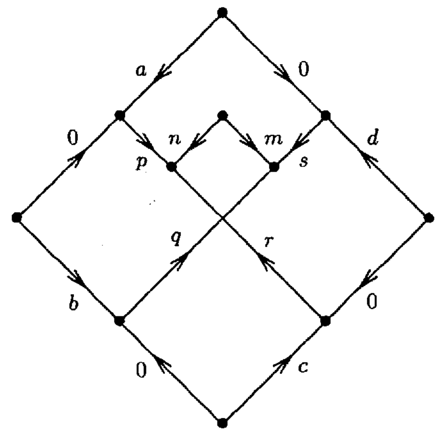

Fig. 1

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix},$$

if and only if all the following 18 quantities are non-negative: a_1, a_2, c_1, c_2, d, e, $b_1 + c_1$, $b_2 + c_2$, $a_1 + b_1 + e$, $a_2 + b_2 + e$, $b_1 + d + e$, $b_2 + d + e$, $c_1 + c_2 + f$, $d + e + f$, $a_1 + a_2 + e + f$, $a_1 + c_2 + e + f$, $a_2 + c_1 + e + f$, $b_1 + b_2 + d + e$.

The proof is analogous to that for (3.1), except that there is no graph corresponding to that of Figure 1, and so we need a version of Hoffman's theorem for regular matroids [9]. We omit the details. Incidentally, the method of simply eliminating variables one by one also provides a reasonable alternative proof of (3.2); with (3.1), this method seems to be too cumbersome for convenient hand calculation.

4. Piecing together successful graphs

A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G with no common edges and with $G_1 \cup G_2 = G$.

(4.1) *Let G be free, let s_1, $s_2 \in V(G)$ be distinct, and let (G_1, G_2) be a separation of G with $s_1 \in V(G_1)$, $s_2 \in V(G_2)$ and $V(G_1 \cap G_2) = \{t\}$ where $t \neq s_1$, s_2. If (G_1, s_1, t) and (G_2, s_2, t) are both successful then so is (G, s_1, s_2).*

Proof. Let (S_1, S_2) be a reasonable goal for (G, s_1, s_2); we shall show that it is attained. We abbreviate $p(G_1, s_1, t)$ by $p(G_1)$, etc. For $i = 1, 2$ let $d_i(t)$ be the valency of t in G_i.

(1) If there are integers $x, y, z, w \geq 0$ such that

$$\begin{aligned} x &\leq |S_1 - S_2| \\ y &\leq k - |S_1 \cup S_2| \\ z &\leq |S_2 - S_1| \\ w &\leq |S_1 \cap S_2| \\ x - y + z - w &\leq |S_1 \Delta S_2| - d_2(t) \\ -x + y - z + w &\leq k - |S_1 \Delta S_2| - d_1(t) \\ x - y &\leq p(G_2) - k + |S_1 - S_2| \\ y - z &\leq p(G_1) - |S_1 \cup S_2| \\ z - w &\leq q(G_2) - |S_1 \cap S_2| \\ w - x &\leq q(G_1) - |S_1 - S_2| \end{aligned}$$

then (S_1, S_2) is attained.

For choose eight subsets X_1, X_2, Y_1, Y_2, Z_1, Z_2, W_1, W_2 of $\{1, \ldots, k\}$ mutually disjoint and with union $\{1, \ldots, k\}$, such that $X_1 \cup X_2 = S_1 - S_2$, $Y_1 \cup Y_2 = \{1, \ldots, k\} - (S_1 \cup S_2)$, $Z_1 \cup Z_2 = S_2 - S_1$, $W_1 \cup W_2 = S_1 \cap S_2$, and

$$\begin{aligned} |X_1| &= |S_1 - S_2| - x, \\ |X_2| &= x, \\ |Y_1| &= y, \\ |Y_2| &= k - |S_1 \cup S_2| - y, \\ |Z_1| &= |S_2 - S_1| - z, \\ |Z_2| &= z, \\ |W_1| &= w, \\ |W_2| &= |S_1 \cap S_2| - w. \end{aligned}$$

(This is possible since the specified cardinalities are all non-negative and sum correctly.) Let $T_1 = X_1 \cup Y_1 \cup Z_1 \cup W_1$, $T_2 = X_2 \cup Y_2 \cup Z_2 \cup W_2$. We claim that for $i = 1, 2$, (S_i, T_i) is a reasonable goal for (G_i, s_i, t). For

$$\begin{aligned} |S_1 \cup T_1| &= |S_1 \cup Y_1 \cup Z_1| = |S_1| + y + |S_2 - S_1| - z \leq p(G_1), \\ |S_1 \cap T_1| &= |X_1 \cup W_1| = |S_1 - S_2| - x + w \leq q(G_1), \\ |S_2 \cup T_2| &= |S_2 \cup X_2 \cup Y_2| = |S_2| + x + k - |S_1 \cup S_2| - y \leq p(G_2), \\ |S_2 \cap T_2| &= |Z_2 \cup W_2| = z + |S_1 \cap S_2| - w \leq q(G_2), \\ |T_1| + d_1(t) &= |S_1 - S_2| - x + y + |S_2 - S_1| - z + w + d_1(t) \leq k, \\ |T_2| + d_2(t) &= x + k - |S_1 \cup S_2| - y + z + |S_1 \cap S_2| - w + d_2(t) \leq k \end{aligned}$$

and $|S_1| + d(s_1), |S_2| + d(s_2) \leq k$ since (S_1, S_2) is a reasonable goal for (G, s_1, s_2). This proves our claim that (S_i, T_i) is a reasonable goal for (G_i, s_i, t) $(i = 1, 2)$. Since G_i is free and (G_i, s_i, t) is successful, it follows that (G_i, s_i, t) attains (S_i, T_i); let κ_i be the corresponding edge-colouring $(i = 1, 2)$. For $e \in E(G)$, define $\kappa(e) = \kappa_i(e)$ where $e \in E(G_i)$. Since $T_1 \cup T_2 = \{1, \ldots, k\}$ it follows that κ is a k-edge-colouring of G, and hence (G, s_1, s_2) attains (S_1, S_2) as required.

From (3.1), (2.1) and (1), to complete the proof it suffices to check that the following 26 inequalities hold:

(a) $|S_1 - S_2| \geq 0$
(b) $k - |S_1 \cup S_2| \geq 0$
(c) $|S_2 - S_1| \geq 0$
(d) $|S_1 \cap S_2| \geq 0$
(e) $k - d_1(t) - d_2(t) \geq 0$
(f) $q(G_1) \geq 0$
(g) $p(G_2) - |S_2| \geq 0$
(h) $p(G_1) - |S_1| \geq 0$
(i) $q(G_2) \geq 0$
(j) $q(G_1) + q(G_2) - |S_1 \cap S_2| \geq 0$
(k) $q(G_1) + p(G_2) - |S_1 \cup S_2| \geq 0$
(l) $p(G_1) + p(G_2) - |S_1 \cap S_2| - k \geq 0$
(m) $p(G_1) + q(G_2) - |S_1 \cup S_2| \geq 0$
(n) $k - d_1(t) \geq 0$
(o) $k - d_2(t) \geq 0$
(p) $q(G_2) + k - |S_2| - d_1(t) \geq 0$
(q) $q(G_1) + k - |S_1| - d_2(t) \geq 0$
(r) $p(G_2) - d_1(t) \geq 0$
(s) $p(G_1) - d_2(t) \geq 0$
(t) $p(G_2) + q(G_2) - |S_2| - d_1(t) \geq 0$
(u) $p(G_1) + q(G_1) - |S_1| - d_2(t) \geq 0$
(v) $p(G_1) + q(G_1) + q(G_2) - |S_1| - |S_2| \geq 0$
(w) $q(G_1) + p(G_2) + q(G_2) - |S_1| - |S_2| \geq 0$
(x) $p(G_1) + q(G_1) + p(G_2) - |S_1| - k \geq 0$
(y) $p(G_1) + p(G_2) + q(G_2) - |S_2| - k \geq 0$
(z) $p(G_1) + q(G_1) + p(G_2) + q(G_2) - |S_1| - |S_2| - k \geq 0$

Now inequalities (a), (b), (c), (d) are trivial, and (e), (n), (o) hold because $d(t) \leq k$ (since G is free). For (f), (h), (q), (s), (w), (y) it suffices (by (2.3) applied to (G_2, s_2, t) in cases (w), (y)) to show that

$$p(G_1) \geq |S_1|, d_2(t), |S_2| + d(s_2) + d_2(t) - k,$$
$$q(G_1) \geq 0, |S_1| + d_2(t) - k, |S_1| + |S_2| + d(s_2) + d_2(t) - 2k.$$

But since $d(s_1) + |S_1| \leq k$, $d(s_2) + |S_2| \leq k$ and $d_1(t) + d_2(t) \leq k$ it follows from (2.2) that

$$p(G_1) \geq k - d(s_1) \geq |S_1|,$$
$$p(G_1) \geq k - d_1(t) \geq d_2(t) \geq d_2(t) + |S_2| + d(s_2) - k,$$

and from (2.4) that $q(G_1) \geq 0$ and

$$q(G_1) \geq k - d_1(t) - d(s_1) \geq d_2(t) + |S_1| - k \geq |S_1| + |S_2| + d(s_2) + d_2(t) - 2k.$$

This proves (f), (h), (q), (s), (w), (y), and (g), (i), (p), (r), (v), (x) follow by symmetry. For (t),(u) and (z) it suffices by (2.3) to show that

$$2k - d(s_2) - d_1(t) - d_2(t) - |S_2| \geq 0,$$
$$2k - d(s_1) - d_1(t) - d_2(t) - |S_1| \geq 0,$$
$$3k - d(s_1) - d(s_2) - d_1(t) - d_2(t) - |S_1| - |S_2| \geq 0$$

and these follow since $d(s_i) + |S_i| \leq k$ $(i = 1,2)$ and $d_1(t) + d_2(t) \leq k$. Finally, for inequalities (j), (k), (l), (m), choose P_i, $Q_i \subseteq V(G_i)$ with $s_i, t \in P_i, Q_i$ and with P_i even and Q_i odd such that $p(G_i) = \delta(P_i)/2$ and $q(G_i) = (\delta(Q_i) - k)/2$. (The following proof may easily be adapted to handle the cases where $|V(G_1)| = 2$ or $|V(G_2)| = 2$, and we omit those details.) Then $P_1 \cup P_2$, $Q_1 \cup Q_2$ are odd, and $P_1 \cup Q_2$, $P_2 \cup Q_1$ are even, and so

$$\frac{1}{2}\delta(P_1 \cup Q_2), \frac{1}{2}\delta(P_2 \cup Q_1) \geq p(G, s_1, s_2) \geq |S_1 \cup S_2|$$
$$\frac{1}{2}(\delta(P_1 \cup P_2) - k), \frac{1}{2}(\delta(Q_1 \cup Q_2) - k) \geq q(G, s_1, s_2) \geq |S_1 \cap S_2|.$$

But $\delta(P_1 \cup P_2) = \delta(P_1) + \delta(P_2) - k$ and so

$$p(G_1) + p(G_2) = \frac{1}{2}\delta(P_1) + \frac{1}{2}\delta(P_2) = \frac{1}{2}(\delta(P_1 \cup P_2) + k) \geq |S_1 \cap S_2| + k.$$

This proves (l) and (j), (k), (m) follow similarly. This completes the proof. ∎

(4.2) *Let G be free, let s, $t \in V(G)$ be distinct, and let (G_1, G_2) be a separation of G with $V(G_1 \cap G_1) = \{s, t\}$. If (G_1, s, t) and (G_2, s, t) are both successful then so is (G, s, t).*

Proof. Let (S, T) be a reasonable goal for (G, s, t). Let $d_i(s)$, $d_i(t)$ be the valencies of s, t in G_i $(i = 1, 2)$. Again we abbreviate $p(G_1, s, t)$ by $p(G_1)$, etc.
(1) If there are integers $x_1, x_2, y_1, y_2, z \geq 0$ such that

$$x_1 \leq |S - T|,$$
$$x_2 \leq |T - S|,$$
$$-x_1 - y_1 - z \leq k - p(G_2) - d_2(t),$$
$$-x_2 - y_2 - z \leq k - p(G_2) - d_2(s),$$
$$x_1 + y_1 + z \leq p(G_2) - |T| - d_1(t),$$
$$x_2 + y_2 + z \leq p(G_2) - |S| - d_1(s),$$
$$x_1 + x_2 + z \leq q(G_1) + p(G_2) - k - |S \cap T|,$$
$$y_1 + y_2 + z \leq p(G_1) + p(G_2) - k - |S \cup T|,$$
$$-x_1 - x_2 - y_1 - y_2 - z \leq q(G_2) - p(G_2)$$

then (S, T) is attained.

For choose $X_1 \subseteq S - T$ with $|X_1| = x_1$, and $X_2 \subseteq T - S$ with $|X_2| = x_2$; and choose Y_1, Y_2, $Z \subseteq \{1, \ldots, k\} - (S \cup T)$ mutually disjoint, with $|Y_1| = y_1$, $|Y_2| = y_2$,

$|Z| = z + k - p(G_2)$. This is possible since these cardinalities are all non-negative (for $p(G_2) \leq k$ by (2.1)) and

$$y_1 + y_2 + z + k - p(G_2) \leq p(G_1) - |S \cup T| \leq k - |S \cup T|$$

by (2.1). We define

$$\begin{aligned} S_1 &= S \cup X_2 \cup Y_2 \cup Z \\ S_2 &= \{1, \ldots, k\} - (X_2 \cup Y_2 \cup Z) \\ T_1 &= T \cup X_1 \cup Y_1 \cup Z \\ T_2 &= \{1, \ldots, k\} - (X_1 \cup Y_1 \cup Z). \end{aligned}$$

We claim that for $i = 1, 2, (S_i, T_i)$ is a reasonable goal for (G_i, s, t). For

$$\begin{aligned} |S_1 \cup T_1| &= |S \cup T| + y_1 + y_2 + z + k - p(G_2) \leq p(G_1) \\ |S_1 \cap T_1| &= |S \cap T| + x_1 + x_2 + z + k - p(G_2) \leq q(G_1) \\ |S_1 \cup T_2| &= |\{1, \ldots, k\} - Z| = k - z - k + p(G_2) \leq p(G_2) \end{aligned}$$

$$\begin{aligned} |S_2 \cap T_2| &= |S \cap T| + |S - T - X_1| + |T - S - X_2| + |\{1, \ldots, k\} - S \cup T \cup Y_1 \cup Y_2 \cup Z| \\ &= |S \cap T| + |S - T| - x_1 + |T - S| - x_2 + k - |S \cup T| - y_1 - y_2 - z - k + p(G_2) \leq q(G_2) \end{aligned}$$

$$\begin{aligned} d_1(s) + |S_1| &= d_1(s) + |S| + x_2 + y_2 + z + k - p(G_2) \leq k \\ d_1(t) + |T_1| &= d_1(t) + |T| + x_1 + y_1 + z + k - p(G_2) \leq k \\ d_2(s) + |S_2| &= d_2(s) + k - x_2 - y_2 - z - k + p(G_2) \leq k \\ d_2(t) + |T_2| &= d_2(t) + k - x_1 - y_1 - z - k + p(G_2) \leq k. \end{aligned}$$

This proves that for $i = 1, 2$ (S_i, T_i) is a reasonable goal for (G_i, s, t). Since (G_i, s, t) is successful it attains this goal; let κ_i be the corresponding edge-colouring. For $e \in E(G)$ let $\kappa(e) = \kappa_i(e)$ where $e \in E(G_i)$. Since $S_1 \cup S_2 = \{1, \ldots, k\}$ and $T_1 \cup T_2 = \{1, \ldots, k\}$ it follows that κ is a k-edge-colouring of G; and since $S \subseteq S_1 \cap S_2$ and $T \subseteq T_1 \cap T_2$ it follows that (S, T) is attained, as required.

From (3.2), (2.1) and (1), to complete the proof it suffices to check that the following 18 inequalities hold:

(a)	$\lvert S - T\rvert \geq 0$
(b)	$\lvert T - S\rvert \geq 0$
(c)	$p(G_2) - \lvert T\rvert - d_1(t) \geq 0$
(d)	$p(G_2) - \lvert S\rvert - d_1(s) \geq 0$
(e)	$q(G_1) + p(G_2) - k - \lvert S \cap T\rvert \geq 0$
(f)	$p(G_1) + p(G_2) - k - \lvert S \cup T\rvert \geq 0$
(g)	$k - \lvert T\rvert - d_1(t) - d_2(t) \geq 0$
(h)	$k - \lvert S\rvert - d_1(s) - d_2(s) \geq 0$
(i)	$p(G_1) - d_2(t) - \lvert T\rvert \geq 0$
(j)	$p(G_1) - d_2(s) - \lvert S\rvert \geq 0$
(k)	$p(G_1) + q(G_1) + p(G_2) - k - d_2(t) - \lvert S\rvert - \lvert T\rvert \geq 0$
(l)	$p(G_1) + q(G_1) + p(G_2) - k - d_2(s) - \lvert S\rvert - \lvert T\rvert \geq 0$
(m)	$p(G_2) + q(G_2) - d_1(s) - d_1(t) - \lvert S\rvert - \lvert T\rvert \geq 0$

$$(n) \qquad p(G_1)+q(G_1)+p(G_2)+q(G_2)-2k-|S|-|T|\geq 0$$
$$(o) \qquad p(G_1)+q(G_2)-k-|S\cap T|\geq 0$$
$$(p) \qquad p(G_1)+p(G_2)+q(G_2)-k-d_1(s)-|S|-|T|\geq 0$$
$$(q) \qquad p(G_1)+p(G_2)+q(G_2)-k-d_1(t)-|S|-|T|\geq 0$$
$$(r) \qquad p(G_1)+q(G_1)-d_2(s)-d_2(t)-|S|-|T|\geq 0$$

Now inequalities (a), (b) are trivial, and (g), (h) hold since (S,T) is a reasonable goal for (G,s,t). By (2.2), $p(G_1)\geq k-d_1(s)\geq d_2(s)+|S|$ (since $d(s)+|S|\leq k$) and so (j) holds. By the symmetry, (c), (d) and (i) hold. By (2.3), since $p(G_1)\geq d_2(s)+|S|$ and $d_1(t)+d_2(t)+|T|\leq k$, it follows that

$$p(G_1)+p(G_2)+q(G_2)\geq 2k-d_2(t)+|S|\geq k+d_1(t)+|S|+|T|$$

and so (q) holds; and (k), (l), (p) follow from the symmetry. For (m), (n), and (r) it suffices by (2.3) to show that

$$2k-d_1(s)-d_1(t)-d_2(s)-d_2(t)-|S|-|T|\geq 0,$$

which is true since $d_1(s)+d_2(s)+|S|\leq k$ and $d_1(t)+d_2(t)+|T|\leq k$. For (e),(f),(o), choose $P_i,Q_i\subseteq V(G_i)$ with $s,t\in P_i,Q_i$ and with P_i even and Q_i odd, such that $p(G_i)=\delta(P_i)/2$ and $q(G_i)=(\delta(Q_i)-k)/2$ $(i=1,2)$. (For by (j), $p(G_1)\geq|S|$, and so (o) holds if $|V(G_2)|=2$, and similarly (e) holds if $|V(G_1)|=2$.) Then $P_1\cup P_2$ is even, and $P_1\cup Q_2$, $Q_1\cup P_2$ are odd, and so

$$|S\cup T|\leq p(G,s,t)\leq \frac{1}{2}\delta(P_1\cup P_2)=\frac{1}{2}(\delta(P_1)+\delta(P_2)-2k)=p(G_1)+p(G_2)-k$$
$$|S\cap T|\leq q(G,s,t)\leq \frac{1}{2}(\delta(P_1\cup Q_2)-k)=\frac{1}{2}(\delta(P_1)+\delta(Q_2)-3k)=p(G_1)+q(G_2)-k$$
$$|S\cap T|\leq q(G,s,t)\leq \frac{1}{2}(\delta(Q_1\cup P_2)-k)=\frac{1}{2}(\delta(Q_1)+\delta(P_2)-3k)=q(G_1)+p(G_2)-k.$$

This verifies (e), (f), (o) and therefore completes the proof. ∎

5. Series–parallel edge-colouring

Now we apply the results of the previous section to prove Theorem (1.4). We need one more lemma.

(5.1) *If $V(G)=\{s,t\}$ and $|E(G)|=1$ then (G,s,t) is successful.*

Proof. Now $p(G)=k-1$, and so if (S,T) is a reasonable goal then $|S\cup T|\leq k-1$. Let $E(G)=\{e\}$ and define $\kappa(e)$ to be some element of $\{1,\ldots,k\}-S\cup T$. Then κ is a k-edge-colouring and so (S,T) is attained. Thus (G,s,t) is successful, as required. ∎

Proof of (1.4). We proceed by induction on $|V(G)|$. Let G (and hence all its subgraphs) be free. If G is not 2-connected the result follows from our inductive hypothesis, and so we may assume that G is 2-connected; and also that $|E(G)|\geq 2$. Let $\mathcal{F}$ be the set of all separations (F,F') of G with $|V(F\cap F')|=2$ and $E(F),E(F')\neq\emptyset$. From the 2-connectivity of G we deduce

(1) *If* $(F, F') \in \mathcal{F}$ *then both* F *and* F' *have paths joining* s *and* t, *where* $V(F \cap F') = \{s, t\}$.

Let $(F, F') \in \mathcal{F}$, with $V(F \cap F') = \{s, t\}$. We shall prove by induction on $|E(F)|$ that (F, s, t) is successful. By (5.1) we may assume that $|E(F)| > 1$.

(2) *If there is a separation* (F_1, F_2) *of* F *with* $V(F_1 \cap F_2) = \{s, t\}$ *and* $E(F_1)$, $E(F_2) \neq \emptyset$ *then* (F, s, t) *is successful.*

For $(F_1, F_2 \cup F'), (F_2, F_1 \cup F') \in \mathcal{F}$, and so from our inductive hypothesis (F_1, s, t) and (F_2, s, t) are successful. Since F is free we deduce from (4.2) that (F, s, t) is successful as required.

(3) *If there do not exist two paths of* F *between* s, t *with no common vertices except* s, t *then* (F, s, t) *is successful.*

For by Menger's theorem there would be a separation (F_1, F_2) of F with $s \in V(F_1), t \in V(F_2)$ and $V(F_1 \cap F_2) = \{u\}$ where $u \neq s, t$. Then $E(F_1)$, $E(F_2) \neq \emptyset$ and F_1, F_2 are free, and so (F_1, s, u) and (F_2, t, u) are successful, from our inductive hypothesis; and hence so is (F, s, t), from (4.1).

To conclude, we prove that one of (2), (3) applies. For suppose not. Then s, t are not adjacent in F, because if they were then since $|E(F)| > 1$ there would be a separation as in (2). Since (3) does not apply ,there are paths P_1, P_2 of F between s, t with no common vertex except s, t. Since s, t are not adjacent in F, P_1 and P_2 both have internal vertices. Since there is no separation as in (2), there is a path P_3 of F from $V(P_1)$ to $V(P_2)$ with $s, t \notin V(P_3)$, and with no vertex in $V(P_1 \cup P_2)$ except its ends. Let P_4 be a path of F' between s and t. Then $P_1 \cup P_2 \cup P_3 \cup P_4$ is a subdivision of K_4, a contradiction. This completes the inductive proof that (F, s, t) is successful, for every $(F, F') \in \mathcal{F}$.

To complete the proof of the theorem, let $f \in E(G)$ with ends s, t, let $F = G \backslash f$, and let F' be the graph with $V(F') = \{s, t\}$, $E(F') = \{f\}$. Now $(F, F') \in \mathcal{F}$ since $|E(G)| \geq 2$, and so (F, s, t) is successful. Let $S = T = \{1\}$. Now if $X \subseteq V(G)$ with $s, t \in X$ then $\delta_F(X) = \delta_G(X) + 2$; and so if X is even then $\delta_F(X) \geq 2$, and if X is odd then $\delta_F(X) \geq \delta_G(X) + 2 \geq k + 2$ since G is free. Hence $p(F, s, t,) \geq 1$ and $q(F, s, t) \geq 1$. Moreover

$$d_F(s) + |S| = d_F(s) + 1 = d_G(s) \leq k$$

and similarly $d_F(t) + |T| \leq k$, and so (S, T) is a reasonable goal for (F, s, t). Since (F, s, t) is successful there is a k-edge-colouring κ of F such that

$$\{\kappa(e) : e \in E(G) \text{ and } e \sim s \text{ or } e \sim t\} \subseteq \{2, \ldots, k\}.$$

Define $\kappa(f) = 1$; then this yields a k-edge-colouring of G, as required. ∎

6. Series–parallel vertex-colouring

We come now to the second result of the paper. This is again about colouring series–parallel graphs, but otherwise is unrelated to what we have done so far, and in particular concerns vertex-colouring, not edge-colouring. By a *k-colouring* of G we mean a function $\kappa : V(G) \to \{1, \ldots, k\}$ such that if $u, v \in V(G)$ are adjacent then $\kappa(u) \neq \kappa(v)$. It is very easy to find a 3-colouring of any series–parallel graph by recursively stripping away vertices with at most two neighbours. There is another simple algorithm which works for series–parallel graphs as we shall see, but curiously its correctness is much less obvious.

Here is an algorithm which attempts to find a k-colouring of a general input graph G.

Algorithm. We recursively colour vertices, one at each iteration. Thus, at the beginning of the first iteration $\kappa(v)$ is not defined for any vertex v. At the beginning of the ith iteration, we denote by X_i the set of all vertices v for which $\kappa(v)$ is defined; thus $|X_i| = i - 1$. For $v \in V(G)$ we define

$$N_i(v) = \{\kappa(u) : u \in X_i \text{ is adjacent to } v\}.$$

The ith iteration proceeds as follows. If $X_i = V(G)$ we announce "success" and stop. Otherwise we choose $v \in V(G) - X_i$ with $|N_i(v)|$ maximum. If $N_i(v) = \{1, 2, \ldots, k\}$ we announce "stuck" and stop. Otherwise we choose j with $1 \leq j \leq k$ and $j \notin N_i(v)$, define $\kappa(v) = j$, and return for the next iteration.

If $k = 2$ and G is bipartite then the algorithm always succeeds, but in general it is not much good. For instance, if $k \geq 3$ the algorithm can get stuck even if G is bipartite, as the second example (due to N. Alon) in Figure 2 shows.

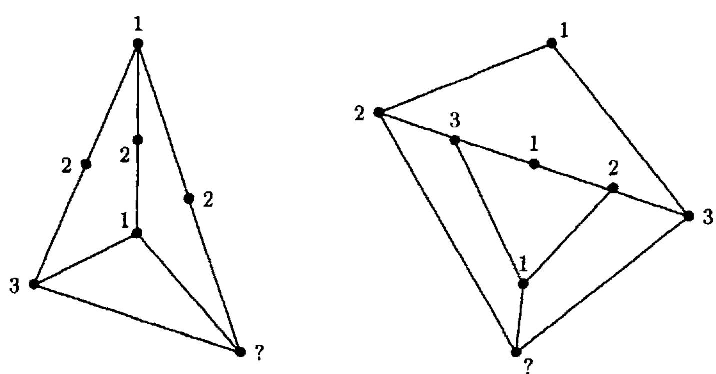

Fig. 2

With $k = 3$, one implementation of the algorithm on either graph from the Figure chooses vertices in vertical order, starting from the top, and defines κ iteratively as shown, and gets stuck at the bottom vertex; and yet both graphs are 3-colourable.

On the other hand, we shall show

(6.1) *If G is series–parallel and k-colourable then the algorithm succeeds.*

Incidentally, to see the motivation for this result, it may help to view it as asserting not that "here is yet another algorithm to colour series–parallel graphs", but rather that "here is a surprisingly rich class of graphs for which our naive algorithm works".

Since the algorithm succeeds if $k \le 2$ and G is k-colourable, we may assume that $k \ge 3$. Thus (6.1) is implied by the following.

(6.2) *Let $n \ge 0$, let $v_1, \dots, v_t$ be distinct vertices of a series–parallel graph G, and let $\kappa : \{v_1, \dots, v_{t-1}\} \to \{1, \dots, n\}$ be such that for $1 \le i \le t$, $|N_i(v_i)| \ge |N_i(v)|$ for each $v \in V(G) - \{v_1, \dots, v_{i-1}\}$, where*

$$N_i(v) = \{\kappa(v_h) : 1 \le h < i,\ v_h \text{ is adjacent to } v\}.$$

Then $|N_t(v_t)| \le 2$.

Proof. By restricting κ to the component containing v_t, we may assume that G is connected. Suppose for a contradiction that $|N_t(v_t)| \ge 3$, with t as small as possible. Then $|N_i(v_i)| \le 2$ for $1 \le i < t$. Define $X_i = \{v_1, \dots, v_{i-1}\}$ for $1 \le i \le t+1$.

(1) *For $1 \le i \le t$ the subgraph of G induced by X_{i+1} is connected.*

To prove this we proceed by induction on i. The result holds if $i = 1$, and we assume that $i \ge 2$. Some vertex $v \in V(G) - X_i$ satisfies $N_i(v) \ne \emptyset$ since G is connected, $X_i \ne \emptyset$ and $X_i \ne V(G)$; and so by hypothesis, $|N_i(v_i)| \ge |N_i(v)| > 0$. Thus v_i is adjacent to a vertex in X_i and the claim follows from our inductive hypothesis.

(2) *v_{t-1} is adjacent to v_t.*

For by hypothesis, $|N_{t-1}(v_t)| \le |N_{t-1}(v_{t-1})| \le 2$ and so $N_{t-1}(v_t) \ne N_t(v_t)$. The claim follows.

Since $|N_t(v_t)| \ge 3$ we may choose three neighbours d_1, d_2, d_3 of v_t with d_1, d_2, $d_3 \in X_t$ and with $\kappa(d_1)$, $\kappa(d_2)$, $\kappa(d_3)$ distinct; and by (2) we may take $d_3 = v_{t-1}$. By (1) there is a vertex v_s where $1 \le s < t$ and three paths P_1, P_2, P_3 of G, mutually disjoint except for v_s, such that P_p has ends v_s, d_p and $V(P_p) \subseteq X_t$ ($1 \le p \le 3$). (Possibly $v_s = d_1$, d_2 or d_3.) Since G is series–parallel there is a partition (Y_1, Y_2, Y_3) of $X_t - \{v_s\}$ such that $V(P_p) \subseteq Y_p \cup \{v_s\}$ ($1 \le p \le 3$) and no edge of G has ends in two different Y_p's.

(3) *For $2 \le k \le t-1$ if for some p ($1 \le p \le 3$) $v_k \in Y_p$, $v_{k-1} \notin Y_p$, and not all $v_1, \dots, v_k \in Y_p \cup \{v_s\}$ then $|N_k(v_k)| \le 1$.*

For we proceed by induction on k. Since not all $v_1, \dots, v_k$ belong to $Y_p \cup \{v_s\}$ we may choose g with $1 \le g \le k-1$ and with $v_g \notin Y_p \cup \{v_s\}$. If none of $v_1, \dots, v_{k-1}$ belong to Y_p then $|N_k(v_k)| \le 1$ as required, and so we may assume that there exists h with $1 \le h \le k-1$ and with $v_h \in Y_p$; and let us choose h maximum with this property. Since $v_{k-1} \notin Y_p$ it follows that $h \le k-2$. If $v_{k-1} \in Y_p \cup \{v_s\}$ then $s = k-1$ and hence g, $h < s$ contrary to (1). Thus we may choose i with $h+1 \le i \le k-1$ such that $v_i \notin Y_p \cup \{v_s\}$; and let us choose i minimum with this property. Let $v_i \in Y_q$ where $1 \le q \le 3$ and $q \ne p$. Now $v_{i-1} \notin Y_q$ from the minimality of i, and not all of $v_1, \dots, v_i$ belong to $Y_q \cup \{v_s\}$ since $h \le i$ and $v_h \in Y_p$. Thus from our inductive hypothesis, $|N_i(v_i)| \le 1$. But $|N_i(v_k)| \le |N_i(v_i)|$ by hypothesis, and so $|N_i(v_k)| \le 1$. If there exists j with $i < j < k$ such that v_j is adjacent to v_k, then $v_j \in Y_p \cup \{v_s\}$; but $j \ne s$ by (1) since h, $i < j$, and $v_j \notin Y_p$ by the maximality of h, a contradiction. Thus there is no such j, and so $N_i(v_k) = N_k(v_k)$ and the claim follows.

To complete the proof, let $d_1 = v_i$, $d_2 = v_j$, where we may assume $i < j$ from the symmetry. Since every path of G between d_1 and d_2 passes through v_s or v_t it follows from (1) that $s \leq j$. Since $j < t-1$ it follows that $v_{t-1} \neq v_s$ and so $v_{t-1} \in Y_3$; and hence we may choose k with $j \leq k \leq t-1$ minimum such that $v_k \in Y_3$. Now $\kappa(d_1)$, $\kappa(d_2) \in N_k(v_t)$ and so $|N_k(v_t)| \geq 2$, and hence $|N_k(v_k)| \geq 2$ by hypothesis. But $k \neq j$ since $v_j \notin Y_3$, and so $v_{k-1} \notin Y_3$ from the minimality of k, and not all $v_1, \ldots, v_k$ belong to $Y_3 \cup \{v_s\}$ since neither v_i nor v_j belongs to Y_3; and this contradicts (3). ∎

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